

HOLOMORPHIC APPROXIMATION ON REAL SUBMANIFOLDS OF \mathbb{C}^{n*}

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§1. This is essentially a summary of results which were announced in [4], and whose proofs are contained in [5].

The general problem here is the following: given a compact set M in a paracompact complex manifold X , to determine those functions on M which can be approximated uniformly on M by functions that are holomorphic on a neighborhood of M . We shall content ourselves with the consideration of the case where M is a C^∞ compact real submanifold of X (with or without boundary), and we will solve the problem for some manifolds of a very particular type. (The differentiability assumption can be weakened, without altering the proofs.)

Let n be the complex dimension of X and let M be a C^∞ submanifold of X of real dimension $2n - k$. Assume that M is given near $p \in M$ by

$$\rho_j = 0; \quad j = 1, \dots, k,$$

where the ρ_j are C^∞ real valued functions whose Jacobian matrix has maximal rank on a neighborhood of p . Let $\{z_1, \dots, z_n\}$ be a system of complex coordinates for X near p . We say that the n -tuple of complex numbers $\{t_i\}$ is a holomorphic tangent vector to M at p if

$$\sum_{i=1}^n t_i \frac{\partial \rho_j}{\partial z_i}(p) = 0 \quad \text{for } j = 1, \dots, k.$$

The set of holomorphic tangent vectors at p is a complex-linear subspace $H_p(M)$ of the tangent space to M at p . We set $m(p)$ = complex dimension of $H_p(M)$ and it is easy to verify that

$$n - k \leq m(p) \leq n - \frac{k}{2}.$$

We will say that M is a CR (Cauchy-Riemann) submanifold of X if $m(p)$ is constant on each connected component of M , and $m(p)$ will be called the CR dimension of M . Now let f be a smooth complex valued function

* Research supported by Army Grant DA-ARO-D-31-124-G866.

defined on a neighborhood of p in the CR submanifold M ; f will be called CR function at p if for a set $\{V_1, \dots, V_{m(p)}\}$ of vector fields such that $\{V_1(q), \dots, V_{m(p)}(q)\}$ is a basis of $H_q(M)$ for q near p , we have that

$$\bar{V}_1 f(q) = \dots = \bar{V}_{m(p)} f(q) = 0 \quad \text{for } q \text{ near } p.$$

It is easy to verify that all these definitions are intrinsic, that is, independent of the choice of the coordinates, the functions ρ , and the basis $\{V_i\}$.

We will say that M is totally real if $m(p) \equiv 0$ on M . On a totally real submanifold any smooth function is of course CR. If M' is totally real, the closure of any relatively compact open set M in M' with a C^∞ boundary will be called a *finite totally real submanifold*.

Examples. 1) Any smooth hypersurface (real codimension 1) M in \mathbb{C}^n is a CR submanifold of CR dimension $n - 1$, and if M is the boundary of a bounded open set $\Omega \subset \mathbb{C}^n$ such that $\mathbb{C}^n - \bar{\Omega}$ is connected, then the CR functions on M are just the traces of functions which are analytic on Ω , and smooth on M (see [2]).

2) Any submanifold of a totally real submanifold is totally real; in particular, any submanifold of the linear subspace of \mathbb{C}^n generated by the real parts of the z_i is totally real.

3) A submanifold of real dimension 1 is always totally real.

4) The distinguished boundary of an analytic polyhedron is totally real.

§2. If K is a compact subset of a complex manifold X , let $A(K)$ denote the uniform closure of the algebra of functions on K which are analytic on a neighborhood of K . If K is a CR submanifold of X , it is easy to see that $A(K)$ is contained in the closure of the algebra of CR functions on K , so that we may expect $A(K)$ to be equal to $C(K)$, the full algebra of complex valued continuous functions on K , when K is totally real. This is indeed the case, and we have the following:

Theorem 1. *Let M be a compact or a finite totally real submanifold of a complex manifold X . Then there is an open Stein manifold $U \subset X$ such that $M \subset U$ and continuous functions on M can be uniformly approximated by functions holomorphic in U . Moreover, C^∞ functions on M can be approximated in the $C^\infty(M)$ topology by functions holomorphic in U .*

It is still an open problem whether CR functions on a CR submanifold M can be approximated by functions which are analytic in a neighborhood of M , but there are some particular cases in which the answer is affirmative, at least locally, and they are contained in the following two theorems.

Theorem 2. *Let M be a real C^∞ hypersurface of an open set $U \subset \mathbb{C}^n$, and let $p \in M$. Then there is a fundamental system of compact neighborhoods $\{K\}$ of p in M such that if $K \in \{K\}$ and f is a CR function on M near K , f can be approximated uniformly by functions holomorphic on a neighborhood of K in \mathbb{C}^n . The same is true for the $C^\infty(M)$ topology.*

Theorem 3. *Let M be a CR submanifold of an open set $U \subset \mathbb{C}^n$, and suppose that the Levi form of M (see [6]) vanishes identically on a neighborhood of p in M . Then there is a fundamental system of compact neighborhoods $\{K\}$ of p in M such that if $K \in \{K\}$ and f is a CR function on M near K , f can be approximated uniformly (or in the $C^\infty(M)$ topology if f is C^∞) by functions holomorphic on a neighborhood of K in \mathbb{C}^n .*

§3. We will now briefly sketch the proof of these theorems. The proof of Theorem 1 was communicated to us by Lars Hörmander, and those of Theorems 2 and 3 are but slight modifications thereof.

As to the proof of Theorem 1, the first step is to observe that under our hypotheses we can construct pseudoconvex domains shrinking down to M . More precisely, if we take a Hermitian metric on X , and if $d(p, M)$ denotes the normal distance from a point $p \in X$ to M , which is defined for p near M , then the domains $T(\varepsilon) = \{p \in X; d(p, M) < \varepsilon\}$ are pseudoconvex for $\varepsilon > 0$ small enough. Henceforth we will assume that $T(\varepsilon_0)$ is embedded in \mathbb{C}^n , for some $\varepsilon_0 > 0$.

Now, obviously it suffices to show that we can approximate C^∞ functions on M by functions holomorphic in U , and the idea is to use the solution of the so-called $\bar{\partial}$ -Neumann problem (see [1] and [2]) to construct the desired holomorphic functions. Namely, suppose $f \in C^\infty(M)$ and take a C^∞ extension \tilde{f} of f to a neighborhood of M in \mathbb{C}^n . Then \tilde{f} is defined on $T(\varepsilon)$ for $\varepsilon > 0$ small enough, and in view of the pseudoconvexity of $T(\varepsilon)$, it is possible to find a C^∞ function v_ε defined on $T(\varepsilon)$ such that

$$(1) \quad \bar{\partial} v_\varepsilon = \bar{\partial} \tilde{f} \text{ on } T(\varepsilon)$$

and moreover,

$$(2) \quad \|v_\varepsilon\|_{T(\varepsilon), 0} \leq C \|\bar{\partial} \tilde{f}\|_{T(\varepsilon), 0},$$

where $\|\cdot\|_{T(\varepsilon), 0}$ stands for the L^2 -norm on $T(\varepsilon)$, and C is a constant independent of ε and of the functions involved. Now $\tilde{f} - v_\varepsilon$ is holomorphic on $T(\varepsilon)$ by (1), and $\tilde{f} - v_\varepsilon|_M - f = v_\varepsilon|_M$.

For a smooth function v on a neighborhood of a compact set $K \subset \mathbb{R}^m$, we set

$$|v|_{K, k} = \sup \{|D^\alpha v(p)|; p \in K, |\alpha| \leq k\},$$

where we use the customary notation for partial differentiation. What we need is to make $|v_\varepsilon|_{M,k}$ arbitrarily small, for a given k . This we can do, as it will presently be shown, by choosing \tilde{f} conveniently and making ε small enough.

In the first place, an algebraic argument similar to the formal part of the proof of Cauchy-Kowalewski's theorem shows that because M is totally real, given any integer $\alpha > 0$, we can find an extension \tilde{f} of f such that $\bar{\partial}\tilde{f} = O(d^\alpha(p, M))$ as $p \rightarrow M$. Now, by Sobolev's lemma we have that for s large enough

$$(3) \quad |v_\varepsilon|_{M,k} \leq C\varepsilon^{-\nu} \|v_\varepsilon\|_{T(\varepsilon),s},$$

where $\|v_\varepsilon\|_{T(\varepsilon),s}$ is the Sobolev s -norm of v_ε on $T(\varepsilon)$, and C and ν are positive constants which do not depend on ε . On the other hand, we have the elliptic estimate

$$(4) \quad \|v_\varepsilon\|_{T(\varepsilon),s}^2 \leq C'\varepsilon^{-\mu} (\|\bar{\partial}v_\varepsilon\|_{T(2\varepsilon),s-1} + \|v_\varepsilon\|_{T(2\varepsilon),0}),$$

where again C' and μ do not depend on ε ; hence by (1), (2), and (3) we finally get

$$|v_\varepsilon|_{M,k} \leq C''\varepsilon^{-(\nu+\mu)} \|\bar{\partial}\tilde{f}\|_{T(2\varepsilon),s-1},$$

and after choosing α , the order of vanishing of $\bar{\partial}\tilde{f}$ on M , conveniently large, we can make the right hand side of the last inequality arbitrarily small by just taking ε small enough.

To complete the proof of Theorem 1, we observe finally that for $0 < \varepsilon_1 \leq \varepsilon_2$, $T(\varepsilon_1)$ is Runge in $T(\varepsilon_2)$ (for ε_1 and ε_2 small enough), which means that we can approximate holomorphic functions on $T(\varepsilon_1)$ by functions holomorphic on $T(\varepsilon_2)$ uniformly on compact subsets of $T(\varepsilon_1)$. Q.E.D.

For the proof of Theorem 2, we remark that one can find a C^∞ real valued function ρ such that $M = \{\rho = 0\}$ and $\text{grad } \rho \neq 0$ on M , and such that if f is a CR function on M near p , then there is a smooth extension \tilde{f} to a neighborhood of M for which, on some neighborhood of p

$$(5) \quad \bar{\partial}\tilde{f} = g\bar{\partial}\rho,$$

where g is in C^{r-1} if $\tilde{f} \in C^r$. We emphasize that this is true only in a neighborhood of p ; globally we can only write

$$\bar{\partial}\tilde{f} = h_0\bar{\partial}\rho + h_1,$$

where h_1 is a $(0,1)$ form vanishing on M . Moreover, as in the proof of Theorem 1, we can pick \tilde{f} so that in (5), g vanishes to any prescribed order on M . Next we pick functions $\zeta_i(x) \in C^\infty(\mathbf{R})$ such that

- (i) $\zeta_t(x) = 1$ if $|x| \leq t$,
 (ii) $\zeta_t(x) = 0$ if $|x| \geq 2t$,
 (iii) $0 \leq \zeta_t(x) \leq 1$ for every x , and
 (iv) $|D^m \zeta_t(x)| \leq Ct^{-m}$ for some constant C ,
 and we define the $(0, 1)$ forms

$$\alpha_t = \zeta_t(\rho) \bar{\partial} u = \zeta_t(\rho) g \bar{\partial} \rho.$$

The forms α_t are $\bar{\partial}$ -closed, and the L^2 -norms of α_t and of their derivatives up to order m can be made arbitrarily small by taking the order of vanishing of g to be sufficiently high, and then taking t small enough. Omitting further details, the approximating analytic functions are constructed by solving the $\bar{\partial}$ -Neumann problem

$$\bar{\partial} v_t = \alpha_t$$

on a pseudoconvex domain where the \bar{f} satisfying (5) is defined, and reasoning as in the proof of Theorem 1. As $\alpha_t \equiv \bar{\partial} \bar{f}$ when we are near M , $\bar{f} - v_t$ is indeed holomorphic near M . Q.E.D.

In Theorem 3, the vanishing of the Levi form implies the existence of a fundamental system of neighborhoods $\{K\}$ of p such that every $K \in \{K\}$ is the intersection of pseudoconvex domains, and therefore we can proceed as in the proof of Theorem 1. Q.E.D.

§4. With the methods outlined above, one can also prove the following approximation theorem:

Theorem 4. *Let $D \subset \subset \mathbb{C}^n$ be such that there is a domain $D' \supset \supset D$ and a C^∞ plurisubharmonic function ϕ defined in D' with*

- (i) $D = \{x \in D'; \phi(x) < 0\}$, and
 (ii) $d\phi \neq 0$ on ∂D .

Then any function holomorphic on D and C^∞ on \bar{D} , can be approximated uniformly on \bar{D} by functions which are analytic on a neighborhood of \bar{D} .

An open problem is to prove a similar theorem for functions holomorphic in D and just continuous up to the boundary.

We refer to [5] for other results in polynomial approximation which follow easily from Theorem 1 above.

§5. Suppose we have a compact set $S \subset X$, and let K be a compact subset of S such that $S - K$ is totally real manifold. Then one could hope to be able to approximate those functions which are continuous on S and whose restriction to K are in $A(K)$, by functions holomorphic on a neighborhood

of S , at least under suitable differentiability assumptions. But this is not the case, as it can be seen from the following example:

$$S = \{x_1^2 + y_1^2 + x_2^2 = 1, y_2 = 0\} \subset \mathbb{C}^2,$$

where $z_j = x_j + iy_j$, $j = 1, 2$. Here $K = \{(0, 1), (0, -1)\}$. Nevertheless, Hörmander and Wermer have proved in a recent paper ([3]) that approximation is indeed possible if we add the condition that K have a neighborhood K_1 in S which is the intersection of domains of holomorphy. With this extra condition, they prove that S itself is the intersection of domains of holomorphy, which makes it possible to carry out the arguments in the proof of Theorem 1.

In the same paper, Hörmander and Wermer use Theorem 1 to prove the following beautiful result:

Theorem ([3], Theorem 5.1). *Let X be a compact set of \mathbb{C}^n , and let R_1, \dots, R_n be in $C^{n+1}(N)$ for some neighborhood N of X in \mathbb{C}^n . Suppose there is a constant $k < 1$ such that*

$$\sum_{i=1}^n |R_i(z) - R_i(z')|^2 \leq k \sum_{i=1}^n |z_i - z'_i|^2$$

when $z, z' \in N$, and where $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then the functions $z_1, \dots, z_n, \bar{z}_1 + R_1(z), \dots, \bar{z}_n + R_n(z)$ restricted to X generate the full algebra of continuous functions $C(X)$.

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